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Anderson localisation on a Cayley tree: a new model with a simple solution

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Abstract. A new model is introduced for wave propagation on a disordered Cayley tree. The model has sufficient simplifying features (related to the absence of time-reversal symmetry and to phase randomisation) that a straightforward study is possible of the probability distribution for eigenstate amplitudes. It is shown that the model supports extended eigenstates at weak scattering and that eigenstates are exponentially localised at strong scattering. Two further distinctions are also shown to be important. The localised phase is insulating if the exponential decay with distance of eigenstate amplitudes is faster than the exponential growth in number of sites with distance from an origin; otherwise eigenfunctions are not square-integrable and the phase is conducting. Correspondingly, in the extended phase there are large amplitude fluctuations unless the correlation length is smaller than the exponential growth rate of site number with distance.

1. Introduction

It is remarkable that, in over 30 years since Anderson's first discussion of localisation of waves by disorder (Anderson 1958), no really elementary mean-field treatment of the mobility edge has emerged. In this paper we describe a relatively straightforward approach that we hope will provide a useful complement to recent solutions of non-linear models (Efetov 1987a) for localisation.

A natural route to a mean-field theory is to study models defined on a Cayley tree, or its infinite-size limit, the Bethe lattice. Previous discussions of localisation in this context can be divided into three categories. Disordered tight-binding models defined on a Bethe lattice were first examined by Abou-Chacra *et al* (1973) and Abou-Chacra and Thouless (1974), who obtained an integral equation for the probability distribution of the single-particle self-energy. The full integral equation is presumably intractable, being non-linear and in two variables, but from it they demonstrated the stability of localised states at strong disorder and estimated the position of the mobility edge. Subsequently, Kunz and Souillard (1983) proved the existence of extended states in the same model and also derived the critical behaviour of the localisation length near the mobility edge. A second approach, initiated by Efetov (1985, 1987a, b; see also Zirnbauer 1986), has been the study on a Bethe lattice of non-linear sigma models for the localisation transition. Essentially complete results have been obtained for the disorder-averaged two-particle Green functions, although the necessary analysis is very elaborate. A third starting point, and the one closest to our own, was suggested by Shapiro (1983), who considered wave transmission through a Cayley tree made by fitting

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together a large number of 'black-box' scatterers. Each scatterer is characterised by a random S-matrix, chosen from a statistical distribution. He concentrated on the reflection coefficient of the tree, showed that there was a transition and calculated the critical behaviour of the conductance.

We introduce a new model, which belongs to the same category as Shapiro's in that it represents wave propagation on a Cayley tree. It is, we believe, the simplest possible example of its kind. The simplicity has two sources. First, time-reversal invariance is maximally broken in the model, in the sense that each of the 'links' or scattering channels that make up the Cayley tree carries flux only in one direction. (To this extent, the model represents electrons moving in an external magnetic field.) Secondly, certain phase shifts appearing in the model are assumed to be random and uniformly, independently distributed. There is a single parameter, playing the same role as energy in tight-binding models, that controls the strength of scattering and hence the nature of eigenstates.

Our approach to studying the model is also novel. It involves testing the sensitivity of energy levels of the system to (spatially) local perturbations. This sensitivity is, in turn, related to the amplitude of the corresponding eigenstate in the perturbed region. We obtain a non-linear integral recursion relation for the probability distribution of eigenstate amplitudes in a sequence of systems of increasing size. At a technical level, the simplifying feature of this integral equation is that it involves only one variable, rather than two (as appear in other studies of more general models). Because of this, it is not difficult to understand enough about the solutions to deduce the nature of eigenstates, as a function of the scattering parameter in the model.

As expected, there is a transition between extended eigenstates at weak scattering and exponentially localised eigenstates at strong scattering. In addition, two further distinctions arise, which are specific to the Cayley tree and have not previously been much emphasised. These occur because the number of sites on a Cayley tree at a given distance from an origin grows exponentially with distance. The rate of exponential growth can be compared with the correlation length in the extended phase, or localisation length in the localised phase. We find that both phases are subdivided. Close to the main transition the correlation length or localisation length is larger than the growth rate of site number, while far from the transition the respective lengths are smaller than the growth rate. The differences have significant physical consequences. Eigenfunctions in the localised phase are square-integrable only if the localisation length is sufficiently short; otherwise the phase is conducting. Correspondingly, there are strong amplitude fluctuations in the extended phase unless the correlation length is smaller than the growth rate of site number. Thus the localisation transition is split into three stages on a Cayley tree. Earlier work has stressed only one of these: the division between normalisable and unnormalisable eigenfunctions.

The remainder of the paper is organised as follows. The model is defined and our approach to calculating properties of eigenstates is outlined in section 2. These ideas are illustrated in section 3 by applying them to a one-dimensional system. The Cayley tree is treated in section 4. In section 5 we summarise briefly our results and discuss qualitatively the functional form of eigenstates.

2. The model

Our model is defined in a way that is one step removed from a specification of the Hamiltonian. The model represents wave propagation on a network with the form of a

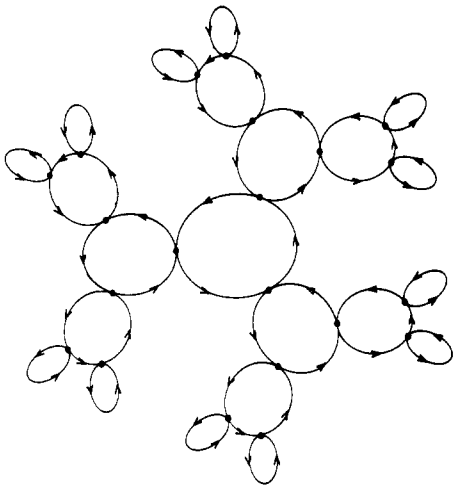

Figure 1. A network having the form of a Cayley tree. Arrows indicate the direction of flux flow on links.

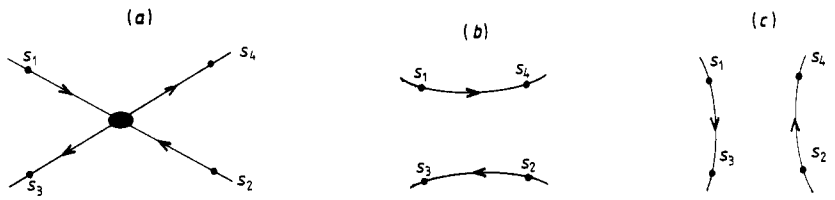

Figure 2. (*a*) A node of the network. The points s_1 to s_4 are referred to in equation (2.2). (*b*) Connections at the node in the limit $\theta = 0$. (*c*) Connections at the node in the limit $\theta = \infty$.

Cayley tree (figure 1). The network is built from two components: 'links', or scattering channels, and nodes, at which links meet and waves are scattered between them. If an underlying Schrödinger equation were solved for every component of the network, the result would be a phase shift characterising each link and a scattering matrix characterising each node. In fact, instead of obtaining these quantities from a Hamiltonian, we define the model directly in terms of phase shifts and scattering matrices. Further simplification comes by restricting each link to carry flux in only one direction, so that time-reversal invariance is broken, as it is for electrons moving in an external magnetic field.

A wave propagating on the network is described at each point s by a complex amplitude, $z(s)$. The phase of $z(s)$ is the same as that of the wave, but the amplitude is scaled according to the group velocity, so that $|z(s)|^2$ gives the flux past the point s. If s_1 and s_2 are the beginning and end of a given link n, then for any $z(s_1)$

$$z(s_2) = \exp(\mathrm{i}\varphi_n)z(s_1) \tag{2.1}$$

where φ_n (real and independent of $z(s_1)$) is the phase shift associated with the link. Four links meet at a node, two carrying incoming fluxes and two carrying outgoing fluxes (figure 2(*a*)). Let $z(s_1)$, $z(s_2)$, $z(s_3)$ and $z(s_4)$ denote the respective amplitudes on each link where it meets the node. For any $z(s_2)$ and $z(s_4)$

$$\begin{bmatrix} z(s_1) \\ z(s_3) \end{bmatrix} = \mathbf{M} \begin{bmatrix} z(s_4) \\ z(s_2) \end{bmatrix} \tag{2.2}$$

in which the (fixed) matrix $\mathbf{M}$ specifies the scattering at the node.

Flux conservation limits **M** to, essentially, a one-parameter family. The condition $|z(s_1)|^2 - |z(s_3)|^2 = |z(s_4)|^2 - |z(s_2)|^2$ for all $z(s_2)$ and $z(s_4)$ implies

$$\mathbf{M} = \begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} e^{i\gamma} & \\ & e^{i\delta} \end{pmatrix} \tag{2.3}$$

with α, β, γ, δ and θ real. The phases α, β, γ and δ can be set to zero by an appropriate choice of gauge, leaving $\theta \geqslant 0$ to parametrise the scattering. We use the notation

$$\mathbf{M}(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{2.4}$$

The limit $\theta = 0$ (figure 2(b)) corresponds to a situation in which all flux incident at the node via s_1 leaves via s_4 (and similarly flux incident via s_2 leaves via s_3). In the opposite limit, $\theta = \infty$ (figure 2(c)), connections are exchanged so that flux incident via s_1 leaves via s_3, etc. More generally, as θ varies between these limits, the scattering amplitudes interpolate between these extremes.

Different models can be obtained by connecting these two components—links and nodes—in various ways. One example of a two-dimensional network has been studied numerically as a model for localisation in the context of the integer quantum Hall effect (Chalker and Coddington 1988), and in fact this setting provided the original motivation for examining systems in which links carry current only in one direction. In the present work we choose networks with a simple enough topology that analytic progress is possible: principally the Cayley tree illustrated in figure 1. This example has coordination $q = 3$, in the sense that the links form loops that (except for those at the surface) are each connected by nodes to three neighbouring loops. It is no more difficult to treat other values of q. Cayley trees with higher coordination show essentially the same behaviour as for $q = 3$, but with extended states over an increasing range as q increases. The case $q = 2$ corresponds to a one-dimensional system. We use it to give a simple demonstration of our methods. In common with other one-dimensional models, it has only localised states.

Having fixed the topology of the network, it remains to specify scattering parameters and phase shifts. We choose the same value for the scattering parameter θ at every node of the network. Changing this value takes the model through its mobility edge. States are extended in the limit $\theta = 0$, when the same flux flows on every link of the system (figure 3(a)), while they are localised for $\theta = \infty$, because then the network decomposes into a large number of uncoupled loops (figure 3(b)).

The link phases cannot be chosen completely arbitrarily, since for a closed network there will not in general be non-zero amplitudes that satisfy equations (2.1) and (2.2) everywhere. This corresponds, of course, to the fact that an arbitrarily chosen energy will not in general be an eigenenergy of a finite system. In order to find energy levels and eigenfunctions, one needs to know something about the variation with energy of the link phases $\{\varphi_n\}$ and the scattering parameter θ. We suppose that the link phases vary much more rapidly with energy than the scattering parameter, which is reasonable if the links are many wavelengths long. Then, in a narrow energy range, we can consider θ fixed and assume φ_n to vary linearly with energy E, i.e.

$$\varphi_n = \alpha_n + \beta_n E. \tag{2.5}$$

Disorder is introduced into the model solely through the choice of α_n. We take α_n for each link n to be an independent random variable, uniformly distributed in $[0, 2\pi)$. It is

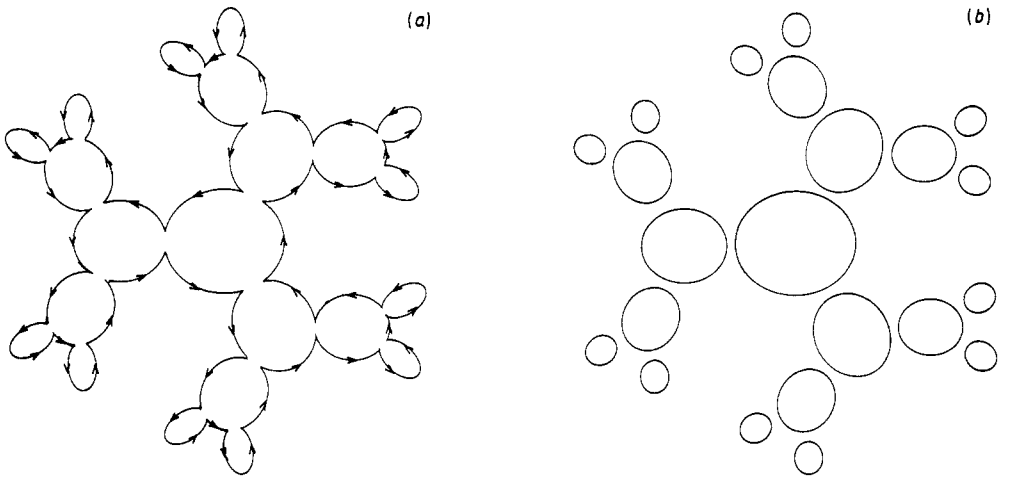

Figure 3. (*a*) The network of figure 1 in the limit $\theta = 0$. (*b*) The network of figure 1 in the limit $\theta = \infty$.

natural, and not difficult, to take β_n also to be a random variable, but for simplicity we choose β_n a constant independent of n, without any important changes in our results.

Solving this model means finding both the allowed energy levels and the associated eigenstates, or flux amplitudes, $z(s)$, which satisfy the matching conditions, equations (2.1) and (2.2), throughout the network. We are interested in statistical properties of these solutions as a function of the scattering parameter θ.

Briefly, our approach is to construct a set of amplitudes $\{z(s)\}$ that satisfy the matching conditions everywhere except possibly at one arbitrarily chosen point, say on the link N. At this point there is, in general, a phase discontinuity of $\Psi(E) + \alpha_N$, where $\Psi(E)$ contains implicit information on the rest of the network and α_N is the random phase associated with this link. Allowed energy levels E_l are solutions to

$$\Psi(E_l) + \alpha_N = 2\pi l \tag{2.6}$$

with l an integer. Thus the function $\Psi(E)$ determines an irregular ladder of energy levels. We find the nature of the associated eigenstates using a method similar to that introduced by Thouless (1974): we test the sensitivity of these energy levels to a local perturbation of the system, represented by a change in the value of α_N. Essentially, this sensitivity, or susceptibility, measures the amplitude of eigenstates in the perturbed region: in fact (see the discussion following equation (3.9))

$$-\mathrm{d}E_l/\mathrm{d}\alpha_N = |z_N|^2 \tag{2.7}$$

where $|z_N|^2$ is the flux carried by the link N. In turn, from equation (2.6), the sensitivity is related to the gradient of $\Psi(E)$ by

$$-(\mathrm{d}E_l/\mathrm{d}\alpha_N)^{-1} = [\partial\Psi(E)/\partial E]|_{E=E_l}. \tag{2.8}$$

One expects the probability distribution for energy level sensitivities to have different characteristic features in regions of localised and extended states, because of the relationship between sensitivities and eigenstate amplitudes. If states are extended, every state should have comparable amplitude at the point where the system is perturbed, and hence each energy level will move by a similar amount under the perturbation. The probability distribution for energy level sensitivities will then be relatively narrow.

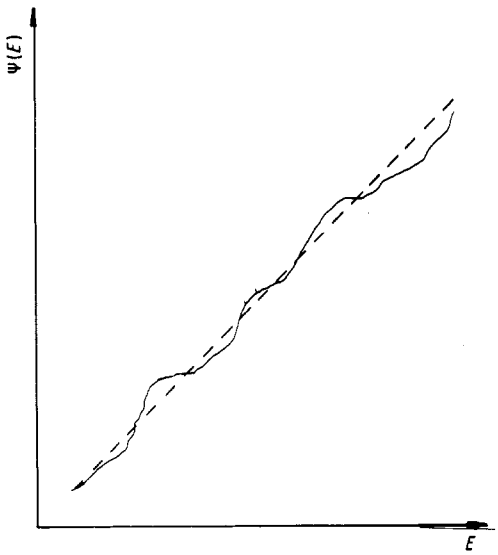

Figure 4. The phase function $\Psi(E)$ versus energy E if states are extended.

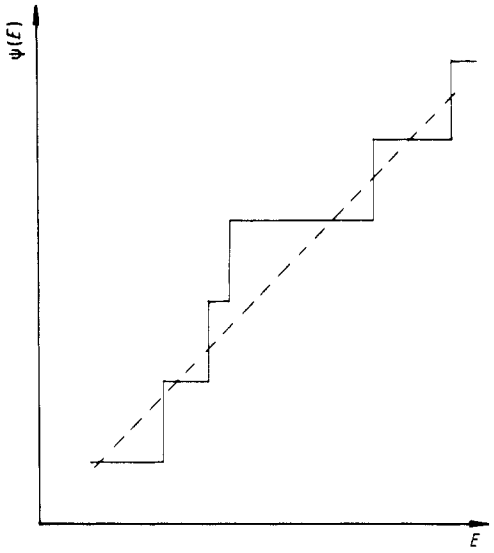

Figure 5. The phase function $\Psi(E)$ versus energy E if states are localised.

Conversely, if states are localised, then, at a given point in the system, a few states will have large amplitudes while almost all others have vanishingly small amplitudes. The probability distribution in that case will be enormously broad.

There are concomitant differences in the function $\Psi(E)$, which can be illustrated if one imagines solving equation (2.6) graphically. Suppose that the function $\Psi(E)$ is plotted against energy E: it is a monotonically increasing function with varying gradient. Energy levels E_l are the energies at which $\Psi(E)$ intersects the horizontal lines: $2\pi l - \alpha_N$. The mean gradient of $\Psi(E)$, averaged over an energy interval large enough to contain many levels, simply determines the density of states: it is proportional to system size and contains no information on localisation. By contrast, fluctuations in the gradient of $\Psi(E)$ are intimately related to the nature of eigenstates. Suppose that $\Psi(E)$ is a smooth function with gradient always close to its mean value (figure 4). Then a perturbation $\delta\alpha$ in the phase α_N will move all energy levels by a similar amount (of order $(\delta\alpha/2\pi) \times$ (mean level spacing)) and we can conclude that states are extended. Suppose, alternatively, that $\Psi(E)$ increases in a sequence of steep steps, with only a small gradient between each rise (figure 5). Then most energy levels (those for which the line $2\pi l - \alpha_N$ intersects a steep part of $\Psi(E)$) will be very insensitive to a perturbation $\delta\alpha$. In that case we conclude that states are localised. Given that states are localised, the argument can be reversed: since most energy levels are then exponentially insensitive (as a function of system size) to the perturbation, the steepness of steps in $\Psi(E)$ should increase exponentially with system size.

In summary, by studying the probability distribution of the phase gradient $\partial\Psi(E)/\partial E$, we obtain information on the probability distribution of fluxes, $|z|^2$, in eigenstates.

3. Localisation in one dimension

In this section we use a one-dimensional network to illustrate the approach outlined above. An attempt is made to construct an eigenstate at energy E on the network of

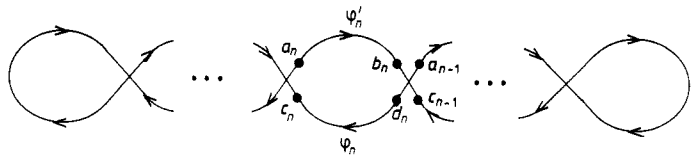

Figure 6. Illustration of the one-dimensional model studied in section 3. The points marked are referred to in equations (3.1*a*) and (3.1*b*).

figure 6 by matching flux amplitudes $z(s)$ across successive links and nodes, working right to left along the length of the network. Whether the result *is* an eigenstate depends on the phase match (or mismatch) $\Psi(E)$ across the final (left-most) link of the network. In turn, the nature of eigenstates is indicated by the phase gradient in energy, $\partial\Psi/\partial E$, which is determined from a stochastic recursion relation (equation (3.8)). An analysis of this recursion relation is the crucial step. It is shown that the phase gradient has a limiting probability distribution in large systems, which, finally, is related to the distribution of eigenstate amplitudes.

Consider the network of figure 6. Using definitions given in section 2 (equations (2.1), (2.2) and (2.4)), amplitudes at successive points are related by

$$\begin{bmatrix} z(b_n) \\ z(d_n) \end{bmatrix} = \mathbf{M}(\theta) \begin{bmatrix} z(a_{n-1}) \\ z(c_{n-1}) \end{bmatrix} \tag{3.1a}$$

and

$$z(a_n) = \exp(-\mathrm{i}\varphi_n')z(b_n) \qquad z(c_n) = \exp(\mathrm{i}\varphi_n)z(d_n). \tag{3.1b}$$

We take $\varphi_n + \varphi_n' = \alpha_n + E$, where the α_n are independent random variables uniformly distributed in $[0, 2\pi)$ and E denotes energy. To find energy levels we need to keep track of the phase difference

$$\psi_n = \arg[z(c_n)] - \arg[z(a_n)]. \tag{3.2}$$

The phase difference ψ_n can be expressed in terms of ψ_{n-1} and α_n in two steps. First, from equation (3.1*a*), we have (using $|z(a_n)| = |z(c_n)|$, a consequence of flux conservation)

$$\arg[z(d_n)] - \arg[z(b_n)] = 2\tan^{-1}[\mathrm{e}^{-2\theta}\tan(\psi_{n-1}/2)] \equiv g(\theta, \psi_{n-1}) \tag{3.3}$$

where the function $g(\theta, \psi)$ is introduced for notational convenience. Secondly, from equation (3.1*b*),

$$\psi_n = \alpha_n + E + g(\theta, \psi_{n-1}). \tag{3.4}$$

In addition, from equation (3.1*a*), fluxes are related by

$$|z(a_{n-1})/z(a_n)|^2 = |\exp(\mathrm{i}\psi_{n-1})\cosh\theta + \sinh\theta|^{-2}$$

which can also be expressed in terms of $g(\theta, \psi)$ as

$$|z(a_{n-1})/z(a_n)|^2 = \partial g(\theta, \psi_{n-1})/\partial\psi_{n-1}. \tag{3.5}$$

The eigenenergies E_l are determined by the phase-matching condition that, in an N-loop chain (see, for example, Ziman 1979),

$$\psi_N(E_l) = 2\pi l. \tag{3.6}$$

(In comparing equations (3.6) and (2.6), note that $\Psi(E) = \psi_N(E_l) - \alpha_N$.)

The function $\psi_N(E)$ is a central quantity in our approach. Its *mean* gradient is simply proportional to the density of states, but its *typical* gradient depends on the nature of eigenstates as outlined in the preceding section. The connection, in detail, is as follows.

From the total derivative, $\mathrm{d}/\mathrm{d}\alpha_N$, of equation (3.6) one finds

$$[\partial\psi_N(E)/\partial E]|_{E=E_l} = -(\mathrm{d}E_l/\mathrm{d}\alpha_N)^{-1}. \tag{3.7}$$

The derivative $\partial\psi_N/\partial E$ satisfies a recursion relation obtained by differentiating equation (3.4), i.e.

$$\frac{\partial\psi_n}{\partial E} = 1 + \frac{\partial g(\theta, \psi_{n-1})}{\partial\psi_{n-1}}\frac{\partial\psi_{n-1}}{\partial E} \tag{3.8}$$

with the initial condition $\partial\psi_1/\partial \mathrm{E} = 1$. Because, at fixed energy, successive ψ_n are uncorrelated, the values of $\partial g(\theta, \psi_{n-1})/\partial\psi_{n-1}$ at each step of the recursion relation are independently distributed. Iterating equation (3.8) and using the relationship between $\partial g(\theta, \psi_{n-1})/\partial\psi_{n-1}$ and ratios of fluxes (equation (3.5)), one finds

$$\partial\psi_N/\partial E = \left(\sum_{n=1}^{N} |z(a_n)|^2\right) \Big/ |z(a_N)|^2. \tag{3.9}$$

It is reasonable to choose $\sum_{n=1}^{N} |z(a_n)|^2 = 1$ for a normalised eigenstate (although normalisation of the probability density itself, rather than the fluxes, would require details of group velocities that have not been specified in the model). With this choice, the sensitivity of an energy level to a perturbation of a given link phase is proportional to the flux carried on that link in the associated eigenstate: $-\mathrm{d}E_l/\mathrm{d}\alpha_N = |z(a_N)|^2$, as anticipated.

The discussion so far has been for $\psi_N(E)$, as a function of energy, in one particular realisation of the random system. It is actually much easier to consider the probability distribution of $\partial\psi_N(E)/\partial E$ at fixed energy, over the ensemble of realisations. Since α_N is uniformly distributed, the probability density for there to be an eigenstate at the chosen energy is simply proportional to $\partial\psi_N(E)/\partial E$. Because of this, it is natural to define two types of average in the ensemble. The average of a quantity X, at fixed energy, over all values of $\{\alpha_n\}$, irrespective of whether the system has an eigenstate at this energy, is denoted by $\langle X\rangle_0$. An average restricted to those systems in the ensemble that have eigenstates at the energy considered is indicated by $\langle X\rangle$, without subscript zero. The two averages are related by

$$\langle X\rangle = \frac{\langle X(\partial\psi_N(E)/\partial E)\rangle_0}{\langle\partial\psi_N/\partial E\rangle_0}. \tag{3.10}$$

As an application of this relationship, one can connect the probability distribution of $\partial\psi_N(E)/\partial E$ to the distribution of eigenstate fluxes through the end link of the chain. If $\rho(x)$ is the probability density for $x \equiv \partial\psi_N(E)/\partial E$ over all systems in the ensemble, and $\mu(y)$ is the probability density for $y \equiv |z(a_N)|^2$ over eigenstates, then one finds from $\mu(y) = \langle\delta(y - [\partial\psi_N(E)/\partial E]^{-1})\rangle$ that

$$\mu(y) = y^{-3}\rho(1/y) \Big/ \int_0^\infty \mathrm{d}x\, x\rho(x). \tag{3.11}$$

We conclude this section by showing from the recursion relation for $\partial\psi_n(E)/\partial E$ (equation (3.8)) that states are indeed localised in the one-dimensional network. Note first, from the definition of $g(\theta, \psi)$ (equation (3.3)), that

$$\left\langle \frac{\partial g(\theta, \psi)}{\partial \psi} \right\rangle_0 = \frac{1}{2\pi} \int_0^{2\pi} [e^{2\theta} \cos^2(\psi/2) + e^{-2\theta} \sin^2(\psi/2)]^{-1} \, d\psi = 1 \tag{3.12}$$

and

$$\left\langle \left(\frac{\partial g(\theta, \psi)}{\partial \psi} \right)^2 \right\rangle_0 = \frac{1}{2\pi} \int_0^{2\pi} [e^{2\theta} \cos^2(\psi/2) + e^{-2\theta} \sin^2(\psi/2)]^{-2} \, d\psi = \cosh(2\theta).$$

Then, by averaging the recursion relation,

$$\langle \partial \psi_N / \partial E \rangle_0 = N \tag{3.13}$$

which simply demonstrates that the mean density of states in energy is proportional to system size. The second moment is more revealing: for large N

$$\left\langle \left(\frac{\partial \psi_N}{\partial E} \right)^2 \right\rangle_0 = N \langle |z(a_N)|^{-2} \rangle \sim \frac{c+1}{(c-1)^2} c^N + O(N) \tag{3.14}$$

with $c = \cosh(2\theta)$. The fact that $\langle (\partial \psi_N / \partial E)^2 \rangle_0$ grows exponentially with N, while $\langle \partial \psi_N / \partial E \rangle_0$ grows only linearly, indicates that $\psi_N(E)$ has the step-like form shown in figure 5. Similarly, the exponential growth of $\langle |z(a_N)|^{-2} \rangle$ suggests that eigenstates are localised: for uniformly extended states, $\langle |z(a_N)|^{-2} \rangle = N$.

A more complete approach is to consider the probability distribution $\rho(x)$ for $x = \partial \psi_N / \partial E$. Since $x \geqslant 1$, one expects either that a limiting distribution exists for large N or that typical values of x diverge with N. If a limiting distribution does exist, then, because $\langle x \rangle_0$ diverges with N, $\rho(x)$ cannot fall more rapidly than x^{-2} as $x \to \infty$. In fact, it is easy to show by applying the inequality

$$(1 + x)^y \leqslant 1 + x^y$$

for $x \geqslant 0$ and $0 \leqslant y \leqslant 1$, to the recursion relation (equation (3.8)), that

$$\langle (\partial \psi_N / \partial E)^y \rangle_0 \leqslant [1 - f(y)^N]/[1 - f(y)]$$

where

$$f(y) \equiv \langle [\partial g(\theta, \psi)/\partial \psi]^y \rangle_0. \tag{3.15}$$

By direct calculation, $0 < f(y) < 1$ for $0 < y < 1$, and hence $\langle (\partial \psi_N / \partial E)^y \rangle_0$ remains bounded as $N \to \infty$. We conclude that a limiting distribution exists, which satisfies the integral equation (from equation (3.8))

$$\rho(x) = \int ds \int dx' \, P(s) \rho(x') \delta(sx' + 1 - x) \tag{3.16}$$

where $P(s)$ is the probability distribution for $s = \partial g(\theta, \psi)/\partial \psi$. The asymptotic behaviour of $\rho(x)$ at large x can be found by substituting the trial form $\rho(x) \sim Ax^{-\alpha}$: consistency requires $\alpha = 2$.

The power law, with $\alpha = 2$, can be understood simply as follows. Suppose eigenstate amplitudes simply decay exponentially from localisation centres that are distributed uniformly along the chain length. Then

$$|z(a_N)|^2 = B e^{-n/\xi} \tag{3.17}$$

where n takes the values $0, 1, \ldots, (N-1)$, each with probability $1/N$, and B and ξ are constants. This defines $\mu(y)$, the probability distribution for $y = |z(a_N)|^2$, as a series of

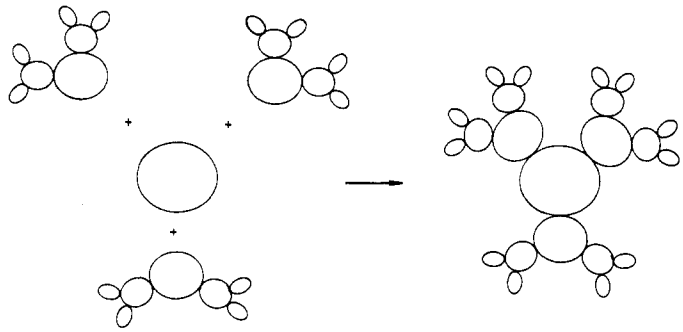

Figure 7. Schematic illustration of the process of constructing a Cayley tree from three trees defined on half-planes.

unevenly spaced delta functions. More realistically, we should expect $\mu(y)$ to be a smoothed version of that function: $\mu(y) = \xi/Ny$ for $B\,e^{-N/\xi} < y < B$, $\mu(y) = 0$ otherwise. From this we deduce (via equation (3.11)) that $\rho(x) \sim x^{-2}$ as $x \to \infty$. In summary, the probability distribution for $\partial\psi_N/\partial E$ is just what one expects from exponentially localised states.

4. The Cayley tree

In this section we apply the approach described above to study the network model defined on a Cayley tree with three nearest neighbours to each loop (figure 1). We examine the probability distribution for eigenstate amplitudes at the centre of the system and identify four regimes of behaviour as the scattering parameter θ is varied. These have, in order decreasing θ: (i) $\infty > \theta > \theta_1$, exponentially localised states with a sufficiently short localisation length that they are normalisable; (ii) $\theta_1 > \theta > \theta_2$, exponentially localised states that are not normalisable, because the localisation length is too large (decay of the eigenfunction with distance is beaten by the exponential growth in site number with distance from an origin on the Cayley tree); (iii) $\theta_2 > \theta > \theta_3$, extended states with broad (power-law) amplitude distributions; and (iv) $\theta_3 > \theta > 0$, extended states with narrow amplitude distributions. We interpret the amplitude fluctuations in regime (iii) as arising because the correlation length is larger than the exponential growth rate in site number with distance. Finally, we study transmission of flux by a Cayley tree with open boundary conditions and show that regime (i) is insulating, but that the other regimes are conducting.

It is most convenient, first, to construct recursively three Cayley trees on half-planes, and then to join them as indicated in figure 7. One stage of the recursive procedure is illustrated in figure 8. In analogy with the treatment described for the one-dimensional network, we define

$$\psi_{n-1} = \arg[z(a_1)] - \arg[z(a_2)] \qquad \psi'_{n-1} = \arg[z(b_1)] - \arg[z(b_2)]$$

and

$$\psi_n = \arg[z(c_1)] - \arg[z(c_2)].$$

The counterparts to equations (3.4) and (3.5) are

$$\psi_n = \alpha_n + E + g(\theta, \psi_{n-1}) + g(\theta, \psi'_{n-1}) \tag{4.1}$$

where α_n is a random phase associated with the elementary loop of figure 8, and

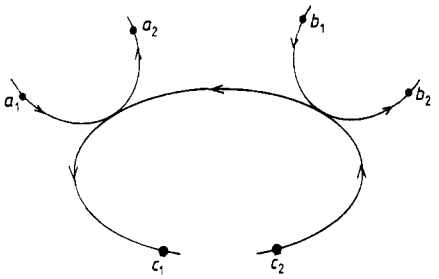

Figure 8. One stage of the recursive procedure represented by equation (4.1).

$$|z(a_1)/z(c_1)|^2 = \partial g(\theta, \psi_{n-1})/\partial \psi_{n-1}. \tag{4.2}$$

Eigenenergies are determined by the condition that phases match at the centre of the tree. With, in obvious notation,

$$\Psi = E + g(\theta, \psi_{N-1}) + g(\theta, \psi'_{N-1}) + g(\theta, \psi''_{N-1}) \tag{4.3}$$

we require

$$\Psi(E) + \alpha_N = 2\pi l$$

(which is equation (2.6)).

As for the one-dimensional model, we shall examine the sensitivity of energy levels to a local perturbation (a change in the value of α_N) and calculate

$$[\partial\Psi(E)/\partial E]^{-1} = -\partial E_l/\partial\alpha_N = |z_N|^2 \tag{4.4}$$

where $|z_N|^2$ is the flux carried by the central loop of the tree. We therefore consider the recursion relation

$$\frac{\partial\psi_n}{\partial E} = 1 + \frac{\partial g(\theta, \psi_{n-1})}{\partial\psi_{n-1}}\frac{\partial\psi_{n-1}}{\partial E} + \frac{\partial g(\theta, \psi'_{n-1})}{\partial\psi'_{n-1}}\frac{\partial\psi'_{n-1}}{\partial E} \tag{4.5}$$

with the initial condition $\partial\psi_1/\partial \mathrm{E} = 1$. Again, progress is possible because ψ_n and ψ'_n are uncorrelated, both mutually and for successive values of n.

The average phase gradient $\langle\partial\Psi/\partial E\rangle_0$ is proportional to the density of states in energy, which is independent of the scattering parameter θ, linear in system size and grows exponentially with the iteration index N:

$$\langle\partial\Psi/\partial E\rangle_0 = 3 \times 2^{N-1} - 2. \tag{4.6}$$

Higher moments of $\partial\Psi/\partial E$ are also easy to calculate but, because of the broad distributions involved, one learns much more by examining $\rho_n(x)$, the probability distribution for $x = \partial\psi_n/\mathrm{d}E$, as a whole. It has four kinds of behaviour with physical interpretations as outlined above. The range of possibilities is reasonably obvious. Since $x \geqslant 1$, either a limiting distribution exists (regime (i)), or typical values of x increase with n. If typical values of x *do* increase with n, one expects, from the multiplicative structure of equation (4.5), that $x \sim \mathrm{O}(a^n)$. The growth of the mean (equation (4.6)) indicates that $a \leqslant 2$. Regime (ii) corresponds to $1 < a < 2$. If $a = 2$, one expects the scaled variable $r = 2^{-n}x$ to have a limiting distribution $\pi(r)$. This may either have a power-law tail for larger r, $\pi(r) \sim r^{-\alpha}$ as $r \to \infty$ (regime (iii)), or fall more rapidly than any power (regime (iv)).

We begin with the largest values of θ: $\infty > \theta > \theta_1$. Then, by an analysis parallel to that preceding equation (3.15), one can show that $\langle(\partial\psi_n/\partial E)^y\rangle_0$ is bounded from above as $n \to \infty$, provided $2f(y) < 1$. The last inequality is satisfied for $y = \frac{1}{2}$ if $\theta > \theta_1$, where

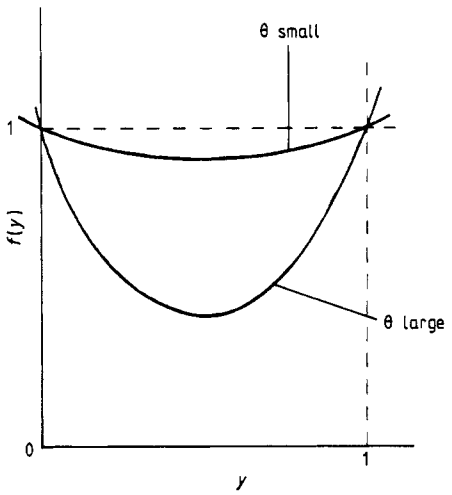

Figure 9. The function $f(y)$, defined in equation (3.15). It can be shown analytically that $f(\frac{1}{2}+y) = f(\frac{1}{2}-y)$ and that $d^2f/dy^2 \geqslant 0$.

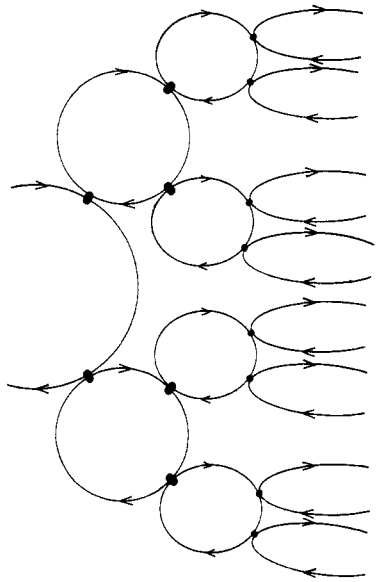

Figure 10. The Cayley tree with open boundary conditions at the surface.

θ_1 is determined below. Hence a limiting distribution $\rho(x)$ exists, satisfying a non-linear integral equation analogous to equation (3.16), i.e.

$$\rho(x) = \int\int\int\int dx_1\, dx_2\, ds_1\, ds_2\, P(s_1)P(S_2)\rho(x_1)\rho(x_2)\delta(x^{-1} - s_1x_1 - s_2x_2)$$

where $P(s)$ is the probability distribution of $s = \partial g(\theta, \psi)/\partial\psi$.

Inserting the trial asymptotic form $\rho(x) \sim Ax^{-\alpha}$, for large x, one finds that the power α is determined by

$$f(\alpha - 1) = \tfrac{1}{2} \tag{4.7}$$

where $f(y)$ is defined in equation (3.15), with α real. The solution to equation (4.7) is best understood graphically; the function $f(y)$ is illustrated in figure 9. For $\theta = \infty$, $\alpha = 2$; α decreases with θ; for $\theta = \theta_1$, $\alpha = \frac{3}{2}$; and for $\theta < \theta_1$, there is no physical solution. By numerical calculation of $f(y)$ we obtain $\theta_1 = 1.88$.

We can relate this behaviour to the nature of eigenstates, as we did for the one-dimensional model. Suppose that all eigenstates have probability densities that simply decay as an exponential, $Be^{-n/\xi}$, with distance n from some centre. These states are normalisable provided the localisation length is sufficiently short: provided $2e^{-1/\xi} < 1$. Assume that the centres of these states are uniformly distributed on the Cayley tree. Then $y \equiv |z_N|^2$ takes the discrete values $y = B\,e^{-n/\xi}$, $n = 0, 1, \ldots, N-1$, with probabilities 2^{n-N}. A smoothed version of this probability distribution is $\mu(y) \propto 2^{-N}y^{-\beta}$ for $B > y > B\,e^{-(N-1)/\xi}$, $\mu(y) = 0$ otherwise, with $\beta = \xi \ln(2) + 1$. Using the connection between $\mu(y)$ and the probability density $\rho(x)$, for the phase gradient $x = \partial\psi_n/\partial E$, we find $\rho(x) \sim x^{-\alpha}$ with $\alpha = 2 - \xi \ln 2$. In the strong scattering limit, $\theta = \infty$, we expect $\xi = 0$ and hence $\alpha = 2$, in agreement with the solution to equation (4.7). Furthermore, we expect ξ to increase as the scattering parameter θ decreases, so that α decreases with θ, as calculated from equation (4.7). However, the simple approximation that all eigenstates

have identical envelopes fails near the mobility edge $\theta = \theta_1$, where states are no longer normalisable. There, the effect of fluctuations, included in equation (4.7), results in the value $\alpha = \frac{3}{2}$, and not $\alpha = 1$ as we might have anticipated from setting $2e^{-1/\xi} = 1$.

The second regime of behaviour, $\theta_1 < \theta < \theta_2$, corresponds to eigenstates that are exponentially localised, but not normalisable. In this case $\partial\psi_n/\partial E$ has no limiting distribution for large n; instead, typical values increase exponentially with n. The rate of exponential growth can be found by studying a modified version of equation (4.5). Consider t_n, which satisfies

$$t_n = 1 + b^{-1}\left(\frac{\partial g(\theta_1\psi_{n-1})}{\partial\psi_{n-1}}t_{n-1} + \frac{\partial g(\theta_1\psi'_{n-1})}{\partial\psi'_{n-1}}t'_{n-1}\right) \tag{4.8}$$

with $t_1 = 1$. Comparing equations (4.8) and (4.5), one sees that $t_n > b^{-n}\partial\psi_n/\partial E$. (It is easier to examine the distribution of t_n than that of $b^{-n}\partial\psi_n/\partial E$ directly, because t_n is bounded away from zero.) If $\partial\psi_n/\partial E \sim O(a^n)$, then t_n will have a limiting distribution as $n \to \infty$, so long as $b > a$. This distribution can be discussed using the same techniques as were applied to equation (4.5); it has the asymptotic form $At^{-\alpha}$ for large t, with α determined by

$$f(\alpha - 1) = \tfrac{1}{2}b^{\alpha-1}. \tag{4.9}$$

We are interested in the smallest value of b for which this equation has a real solution, $\alpha > 1$, and take this value of b to be the mean rate of growth a of $\partial\psi_n/\partial E$. Imagine a graphical solution to equation (4.9). Referring to figure 9, a increases with decreasing scattering θ from $a = 1$ at $\theta = \theta_1$ to $a = 2$ at $\theta = \theta_2$. We find $\theta_2 = \ln(\sqrt{2} + 1) \simeq 0.881$.

Such behaviour can also be related to the form of eigenstates. Typical values of $\partial\Psi/\partial E$ determine the largest values of $|z_N|^2$, which arise from eigenstates having their localisation centre near the middle of the Cayley tree. Assuming non-normalisable, exponential decay, one calculates $|z_N|^2 \sim O(\exp[N(1/\xi - \ln 2)])$ for these states. Thus we identify, using equation (4.4), $a = 2e^{-1/\xi}$. The calculated variation of a with the scattering parameter indicates that the localisation length ξ increases with decreasing θ, and diverges at $\theta = \theta_2$.

The third regime of behaviour, $\theta_2 > \theta > \theta_3$, corresponds to eigenstates that typically have a probability density at the centre of the Cayley tree of order (system size)$^{-1}$, but with a distribution of values that has a power-law tail. We interpret the power-law distribution as indicating that the correlation length of these extended eigenstates is larger than the growth rate of site number with distance, so that amplitude fluctuations are, in a sense, non-normalisable. Consider the recursion relation for $r \equiv 2^{-n}(\partial\psi_n/\partial E)$, which at large n is asymptotically

$$r_n = \tfrac{1}{2}\left(\frac{\partial g(\theta, \psi_{n-1})}{\partial\psi_{n-1}}r_{n-1} + \frac{\partial g(\theta_1, \psi'_{n-1}}{\partial\psi'_{n-1}}r'_{n-1}\right). \tag{4.10}$$

In this regime r has a limiting distribution $\pi(r)$, with asymptotic behaviour $\pi(r) \sim Ar^{-\alpha}$ for large r. Following the same methods as before, α is determined from

$$f(\alpha - 1) = 2^{\alpha-2}. \tag{4.11}$$

A graphical analysis, based on figure 9, indicates that α increases with decreasing scattering, from $\alpha = 2$ at $\theta = \theta_2$ to $\alpha = \infty$ at $\theta = \theta_3$. We find $\theta_3 = \frac{1}{2}\ln 2 \simeq 0.347$. These conclusions can be checked by explicit calculation of integer moments of r, obtained by averaging powers of the recursion relation (equation (4.10)).

Finally, in the fourth regime, $\theta_3 > \theta > 0$, r has a limiting distribution $\pi(r)$ that falls more rapidly to zero with increasing r than any negative power of r. One can confirm directly that all positive integer moments of r are finite.

The discussion so far has concentrated on the distribution of eigenstate amplitudes at a single point in the system. It is in fact the main weakness of our approach that we are not able to calculate two-point correlation functions, such as $\langle |z(a_l)z(a_k)|^2 \rangle$. It is, however, possible to derive a recursion relation for the reflection coefficient from a Cayley tree with open boundary conditions at the surface sites (figure 10). We find that the reflection coefficient iterates to unity if the eigenstates are normalisable ($\infty > \theta > \theta_1$), but otherwise ($\theta_1 > \theta > 0$) has a non-trivial limiting distribution. The arguments, which are similar to those of Shapiro (1983), are as follows.

The reflection coefficient is defined by, referring to figure 8,

$$R_n = |z(c_1)/z(c_2)|^2. \tag{4.12}$$

This can be written, using equation (3.1*a*), as

$$R_n = F(R_{n-1}, \psi_{n-1})F(R'_{n-1}, \psi'_{n-1}) \tag{4.13}$$

where $R_{n-1} = |z(a_1)/z(a_2)|^2$, $R'_{n-1} = |z(b_1)/z(b_2)|^2$, ψ_{n-1} and ψ'_{n-1} are independently and uniformly distributed phases and

$$F(R, \psi) = \frac{\tanh^2 \theta + 2R^{1/2} \tanh \theta \cos \psi + R}{1 + 2R^{1/2} \tanh \theta \cos \psi + R \tanh^2 \theta}. \tag{4.14}$$

If no flux is incident at the surface of the tree, the initial condition for the recursion relation (equation (4.13)) is $R_1 = 0$. To find whether the reflection coefficient iterates to unity, we only need to examine equation (4.13) for $1 - R \equiv \varepsilon \ll 1$. Linearising about $\varepsilon = 0$ and noting that $[\mathrm{d}F(R, \psi)/\mathrm{d}R]|_{R=1}$ can be expressed in terms of the function $g(\theta, \psi)$, we find

$$\varepsilon_n = \frac{\partial g(\theta, \psi_{n-1})}{\partial \psi_{n-1}} \varepsilon_{n-1} + \frac{\partial g(\theta, \psi'_{n-1})}{\partial \psi'_{n-1}} \varepsilon'_{n-1}. \tag{4.15}$$

From our study of equation (4.5), we know that typical values of ε_n iterate to zero (i.e. perfect reflection) if $\theta > \theta_1$, while for $\theta_1 > \theta$, typical values grow exponentially with n and a linearised equation is no longer adequate. Thus the system is insulating if $\theta > \theta_1$ and has a finite conductance if $\theta_1 > \theta$.

5. Discussion

We conclude with a brief summary of our results and a qualitative discussion of the form of eigenfunctions on the Cayley tree.

We have studied a model for wave propagation on a disordered Cayley tree that is, in a useful sense, irreducibly simple. We have shown with relatively straightforward techniques that there are four regimes of behaviour as a scattering parameter in the model is varied. Eigenstates are exponentially localised at strong scattering and extended at weak scattering. In addition, there are important distinctions in behaviour according to whether the localisation length, or correlation length in the extended phase, is larger than the exponential growth rate of site number with distance from an origin on the Cayley tree. Our results supplement the more technical approaches of Kunz and Souillard (1983), Efetov (1985, 1987a, b) and Zirnbauer (1986). The present methods have the weakness that conventional correlation functions of eigenstate amplitudes at two separate points are not accessible; their strengths are simplicity and the fact that they treat probability distributions as a whole, rather than simply averages.

The main feature of our results that is new, at least in emphasis, is the significance of distinctions other than the one (relevant near $\theta = \theta_1$) between square-integrable and non-normalisable eigenstates. The correlation functions calculated by Kunz and Souillard (1983) and Efetov (1987a, b) decay exponentially with separation at all energies, and those authors concentrate on whether correlation functions summed over sites at all separations are finite. To explain better the relationship between our results and theirs, we finish by describing a way of picturing typical eigenfunctions as a whole.

The process of constructing an eigenstate, as discussed in the previous section, involves two stages. First, at an arbitrarily chosen energy, amplitudes and phases are matched at all points on the network except (possibly) one. Secondly, the probability density for the chosen energy actually to be an eigenenergy (in the ensemble of values for α_N; equation (2.6)) is computed to be proportional to $|z_N|^{-2}$ (where $|z_N|^2$ is the flux flowing on the link with which the phase α_N is associated). Thus, at the first stage, an ensemble of possible eigenfunctions is generated, and at the second stage actual eigenfunctions are selected with the correct probability from this ensemble.

The eigenstates are most easily caricatured in the weak scattering regime, $\theta_3 > \theta > 0$, when fluctuations in $|z_N|^2$ are small and all functions constructed in the first stage have similar probabilities actually to be eigenstates. A typical function generated by this first stage is simple to describe. Consider fluxes $|z|^2$ flowing on successive links that form a branch of the Cayley tree leading from the centre to the surface. Let $|z_N|^2$ be the flux on a central link, and $|z_{N-n}|^2$ the flux on a link n steps towards the surface. From equation (3.5)

$$|z_{N-n}|^2 = h_1 h_2 \ldots h_n |z_N|^2 \tag{5.1}$$

where the h_i represent factors $\partial g(\theta, \psi)/\partial \psi$ at the nodes traversed by the path joining the two links. At an arbitrary energy (that is, neglecting the weight determined by $|z_N|^2$), each h_i is statistically independent. Thus, if we imagine $\log|z_{N-n}|$ as a function of n, we obtain a random walk, the increment in the ordinate at the nth step being $\log h_n$. At large scales we know from the central limit theorem that $[\log|z_{N-n}| - \log|z_N|]$ is Gaussian distributed with mean $n\langle \log h_0\rangle$ and variance $n[\langle(\log h)^2\rangle_0 - \langle \log h\rangle_0^2]$. Since $\langle \log h\rangle_0 < 0$ for $\theta > 0$, this means that (for $\theta_3 > \theta > 0$, when the weight involving $|z_N|^2$ does not play a crucial role) eigenstate amplitudes on a typical branch decay exponentially with distance from the centre of the Cayley tree.

It is this exponential decay that is captured by the correlation functions that Kunz and Souillard (1983) and Efetov (1987a, b) study. The story cannot end here: if eigenstate amplitudes decayed exponentially with distance from the centre of the Cayley tree along all branches, the flux $|z_N|^2$ would be O(1). In fact, we know from explicit calculation that (for $\theta_3 > \theta > 0$) $|z_N|^2$ scales inversely with the system size. We conclude that in representative eigenstates there must also be atypical branches of the tree along which the amplitude does not decay. This can happen because of the exponential growth of site number with distance, which means that the tails of the approximately log-normal distribution for $|z_{N-n}|^2$ are important. In summary, this qualitative picture of eigenstates in the extended phase reconciles the exponential decay found by previous authors with our results for eigenstate amplitude distributions.

It is more difficult to obtain much insight from this approach for larger values of the scattering parameter, $\infty > \theta > \theta_3$, but the outline is clear. The central point is that typical functions in the ensemble generated by the first stage, as described above (equation (5.1)), are very different from typical eigenfunctions, as described following equation (4.7). We can understand how this comes about as follows. In this range for θ, values of

$|z|^2$ fluctuate strongly and an arbitrarily chosen energy is *unlikely* to be an eigenenergy. As a result, the values of successive h_i in equation (5.1) are highly correlated at eigenenergies. By way of illustration, consider the insulating phase, $\infty > \theta > \theta_1$. Then, typical eigenstates decay exponentially from a localisation centre. Different localisation centres must be uniformly distributed over the Cayley tree, since the local, disorder-averaged density of states is independent of position. The selection of such eigenstates from the ensemble of functions generated via equation (5.1) is due to the weighting associated with the probability ($\propto |z_N|^{-2}$) for there to be an eigenstate at the energy considered. Most functions in the ensemble have a large value for $|z_N|^2$ and hence a small probability actually to be eigenstates; a small fraction of functions in the ensemble have exponentially small values for $|z_N|^2$, and it is just these that are likely to be eigenstates. Our study of the recursion relation (equation (4.5)) provides a formal treatment of this selection process.

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